LESSON

8 Rates of Change

It is useful to contemplate slopes in practical situations. For example, suppose the following graph in Figure 8.1 is for $y = f(x)$, a function that gives the price y for various amounts x of cheese. Because the straight line goes through the points (1 lb.,\$2) and (2 lbs.,\$4), the slope $= \dfrac{4-2}{2\text{ lbs.} - 1\text{ lb.}} = \dfrac{2}{1\text{ lb.}} = \$2$ per pound.

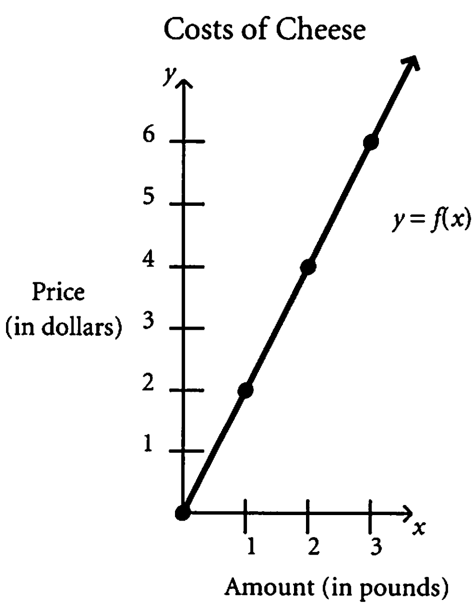


Figure 8.1

The slope is therefore the *rate* at which the cheese is sold, in dollars per pound. Because slope = $\frac{y\text{-}change}{x\text{-}change}$, a slope will always be a rate measured in y-units per x-unit.

For example, suppose a passenger on a bus writes down the exact time she passes each highway mile marker. She then sketches the graph shown in Figure 8.2 of the bus's position on the highway over time. The slope at any point on this graph will be measured in y-units per t-unit, or miles per hour. The steepness of the slope represents the speed of the bus.

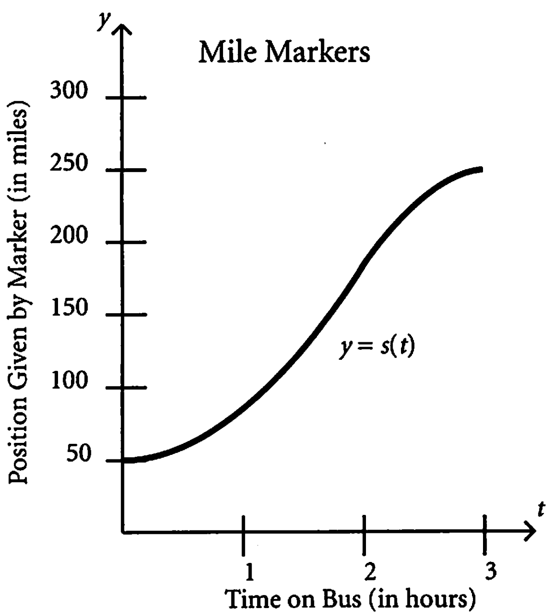
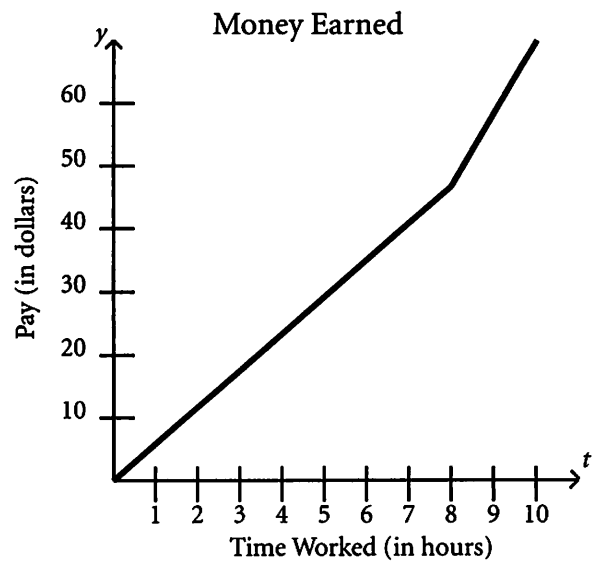


Figure 8.2

▶ Practice

For each of the following four graphs, give the rate that a slope represents.

1.



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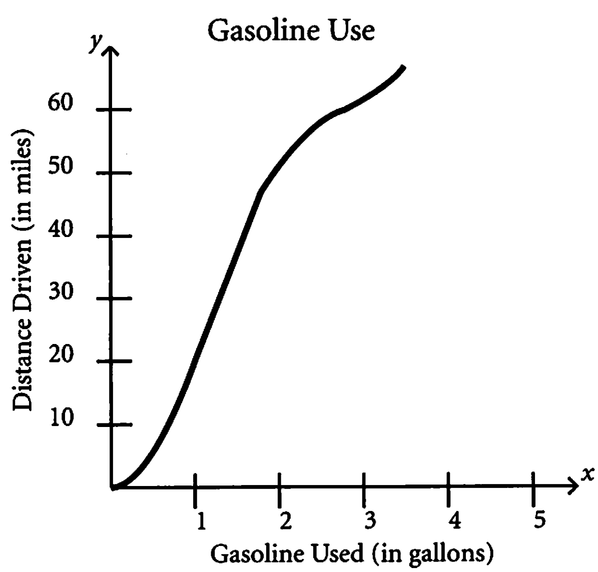

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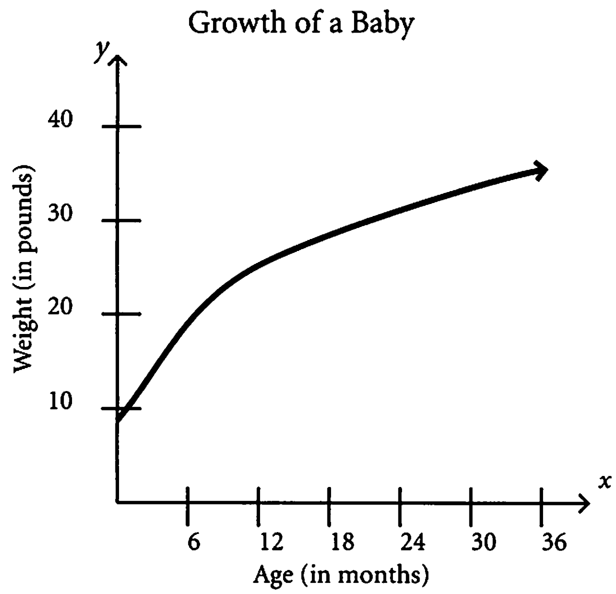

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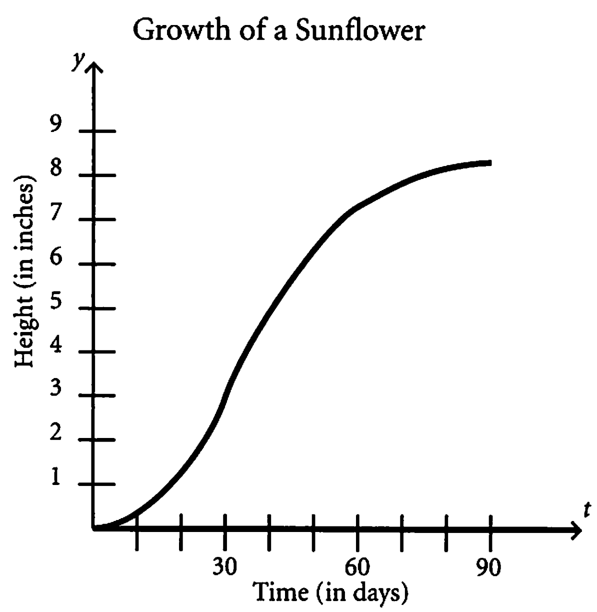

Because the derivative of a function gives the slope of its tangent lines, these practice problems show that the derivative of a function gives its *rate of change.*

An excellent example comes from position functions. A *position function* $s(t)$ states where an object is at any given time. The derivative $s'(t)$ states the rate at which that object's position is changing—that is, the speed or *velocity* of the function. Thus, $s'(t) = v(t)$. The second derivative $s''(t) = v'(t)$ tells how the velocity is changing, or the *acceleration.* Thus, $s''(t) = v'(t) = a(t)$ where $s(t)$ is the position function, $v(t)$ is the velocity function, and $a(t)$ is the acceleration function.

Example

Suppose an object rolls along beside a tape measure so that after t seconds, it is next to the inch marked $s(t) = 4t^2 + 8t + 5$. Where is the object after 1 second? After 3 seconds? What is the velocity function? How fast is the object moving after 2 seconds? What is the acceleration function?

Solution

The position function $s(t) = 4t^2 + 8t + 5$ tells us where the object is. After 1 second, the object is next to the $s(1) = 17$-inch mark on the tape measure. After 3 seconds, the object is at the $s(3) = 65$-inch mark.

The velocity function is $v(t) = s'(t) = 8t + 8$. Thus, after 2 seconds, the object is moving at the rate of $v(2) = 24$ inches per second. Do realize that this velocity of 24 inches per second is an *instantaneous velocity,* the speed just at a single moment. If a car's speedometer reads 60 miles per hour, this does not mean that it will drive for 60 miles or even for a full hour. The car might speed up, slow down, or stop. However, at that instant, the car is traveling at a rate that, if unchanged, will take it 60 miles in one hour. A derivative is always an instantaneous rate, telling you

the slope at a particular point, but not making any promises about what will happen next.

The acceleration function is $a(t) = v'(t) = s''(t) = 8$. Because this is a constant, it tells us that the object increases in speed by 8 inches per second every second.

The most popular example of constant acceleration is gravity, which accelerates objects downward by $32\frac{\text{ft}}{\text{sec}}$ every second. Because of this, an object thrown with a velocity of b feet per second from a height of h feet above the ground will have (after t seconds) a height of $s(t) = -16t^2 + bt + h$ feet.

The starting time is $t = 0$, at which point the object is $s(0) = h$ feet off the ground, the correct initial height. The velocity function is $v(t) = s'(t) = -32t + b$. At the starting time $t = 0$, the velocity is $v(0) = b$, the desired initial velocity. The function $v(t) = -32t + b$ means that 32 feet per second are subtracted from the initial velocity b every second. The acceleration function is $a(t) = v'(t) = s''(t) = -32$. This is the desired constant acceleration.

Example

Suppose a brick is thrown upward at $10\frac{\text{ft}}{\text{sec}}$ from a 150-foot rooftop. What are its position, velocity, and acceleration functions?

Solution

Because the initial velocity is $b = 10\frac{\text{ft}}{\text{sec}}$ and the initial height is $h = 150$ feet, the position function is $s(t) = -16t^2 + 10t + 150$. The velocity function is $v(t) = s'(t) = -32t + 10$. The acceleration is $a(t) = -32$, a constant 32 feet per second downward each second. The negative sign indicates that gravity is acting to decrease the height of the brick, pulling it downward.

Example

Suppose a rock is dropped from a 144-foot tall bridge. When will the rock hit the water? How fast will it be going then?

Solution

Because the rock is dropped, the initial velocity is $b = 0$. The initial height is $h = 144$. Thus, $s(t) = -16t^2 + 144$ gives the height function. The rock will hit the water (have a height of zero) when:

$$-16t^2 + 144 = 0$$

$$144 = 16t^2$$

$$t = \pm 3$$

And because -3 seconds doesn't make any sense, the rock will hit after 3 seconds.

The velocity function is $v(t) = s'(t) = -32t$; therefore, the rock will have a velocity of $v(3) = -96$ after 3 seconds. It will be traveling at a rate of 96 feet per second downward when it hits the water.

Example

If $p(t) = \frac{t^2}{10} - 80t + 50{,}000$ gives the value, in thousands of dollars, of a start-up company after t days, then how fast is its value changing after 30 days? After 500 days?

Solution

The derivative $p'(t) = \frac{t}{5} - 80$ gives the rate of change in value, measured in thousands of dollars per day. After 30 days, $p'(30) = -74$, so the company will be losing \$74,000 of value every day. After 500 days, $p'(500) = 20$, so the company will be gaining value at the instantaneous rate of \$20,000 a day.

▶ Practice

5. The height of a tree after t years is $h(t) = 30 - \frac{25}{t}$ feet when $t \geq 1$. How fast is the tree growing after 5 years?
6. The level of a river t days after a heavy rainstorm is $L(t) = -t^2 + 8t + 26$ feet. How fast is the river's level changing after 7 days?
7. When a company makes and sells x cars, its profit is $P(x) = \frac{x^3}{10} - 60x^2 + 9{,}000x$ dollars. How fast is its profit changing when the company makes 50 cars? Should the company make more cars?
8. When a container is made x inches wide, it costs $C(x) = 0.8x^2 + \frac{24}{x}$ dollars to make. How is the cost changing when $x = 3$ inches? Would it be cheaper to increase or decrease the width?
9. An electron in a particle accelerator is $s(t) = t^3 + 2t^2 + 10t$ meters from the start after t seconds. Where is it after 3 seconds? How fast is it moving then? How fast is it accelerating then?
10. A brick is dropped from 64 feet off the ground. What is its position function? What is its velocity function? What is its acceleration? When will it hit the ground? How fast will it be traveling then?
11. A bullet is fired upward at 800 feet per second from the ground. How high is it when it stops rising and starts to fall?
12. A rock is thrown 10 feet per second down a 1,000-foot cliff. How far has it gone down in the first 4 seconds? How fast is it traveling then?

▶ Derivatives of Sine and Cosine

It is by examining rates and slopes that we can find the derivative of $\sin(x)$. Look at the slopes at various points on its graph in Figure 8.2. It appears that the derivative function of $\sin(x)$ must oscillate between -1 and 1, and must go through the following points (see Figure 8.3). The function $\cos(x)$ is exactly such an oscillating function (see Figure 8.4). This leads us to conclude $\frac{d}{dx}(\sin(x)) = \cos(x)$.

A similar study of the slopes of $\cos(x)$ would show that $\frac{d}{dx}(\cos(x)) = -\sin(x)$. The slopes of the cosine function are not the values of the sine function, but rather their exact negatives.

Examples

Differentiate the following examples.

$f(x) = 5\sin(x) + 4x^2$

$y = 2 + \cos(t)$

$g(x) = \sin(x) - \cos(x)$

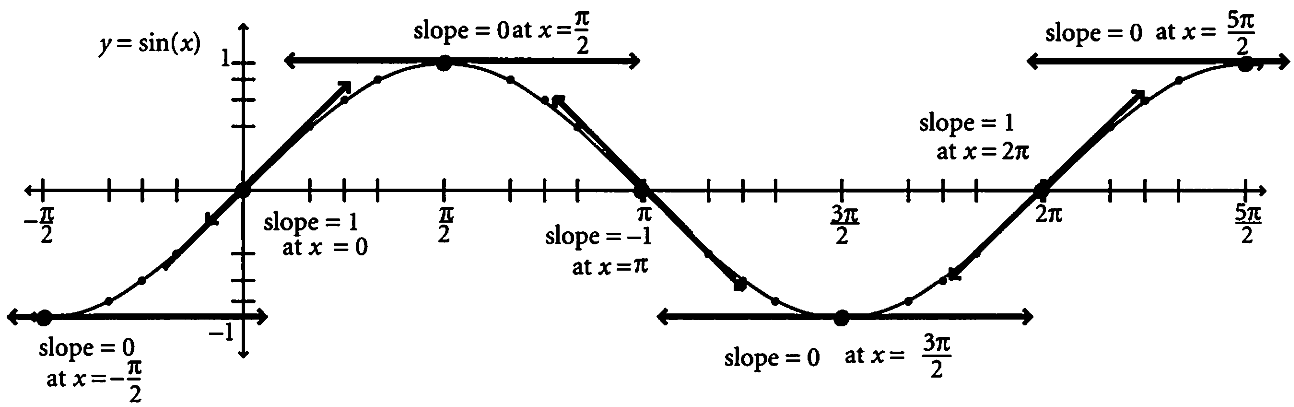

Figure 8.2

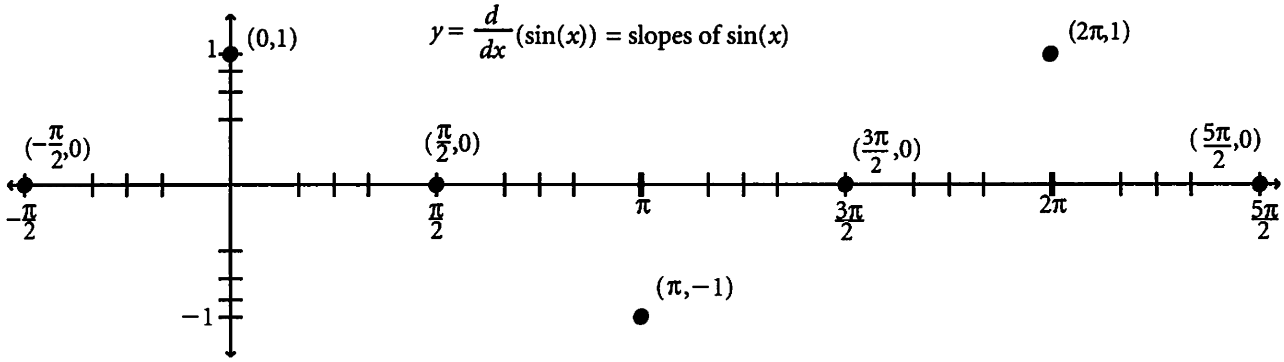

Figure 8.3

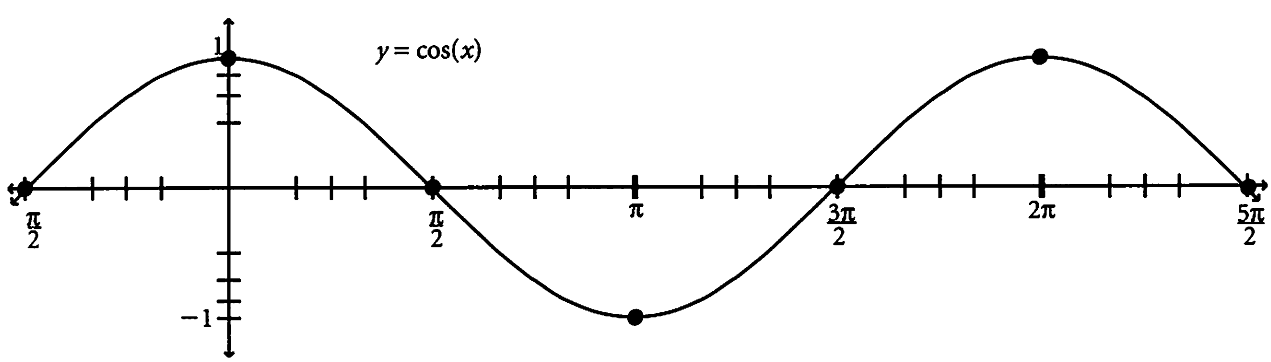

Figure 8.4

Solutions

$f'(x) = 5\cos(x) + 8x$

$\frac{dy}{dt} = -\sin(t)$

$g'(x) = \cos(x) + \sin(x)$

▶ Practice

Differentiate the following practice problems.

13. $y = 4x^5 + 10\cos(x) + 3$

14. $f(t) = 3\sin(t) + \frac{2}{t}$

15. $g(x) = 8x + 3 - \cos(x)$

16. $r(\theta) = \frac{1}{2}\sin(\theta) + \frac{1}{2}\cos(\theta)$

17. $h(x) = \cos(x) + \cos(5)$

18. Find the equation of the tangent line to $f(x) = \sin(x) + \cos(x)$ at $x = \frac{\pi}{2}$.

▶ Derivatives of the Exponential and Natural Logarithm Functions

The reason why the nicest exponential function is e^x where $e = 2.71828\ldots$ is because this makes for the following very nice derivative:

$$\frac{d}{dx}(e^x) = e^x$$

It is only with this exact base that the derivative of the exponential function is itself (see Figure 8.5).

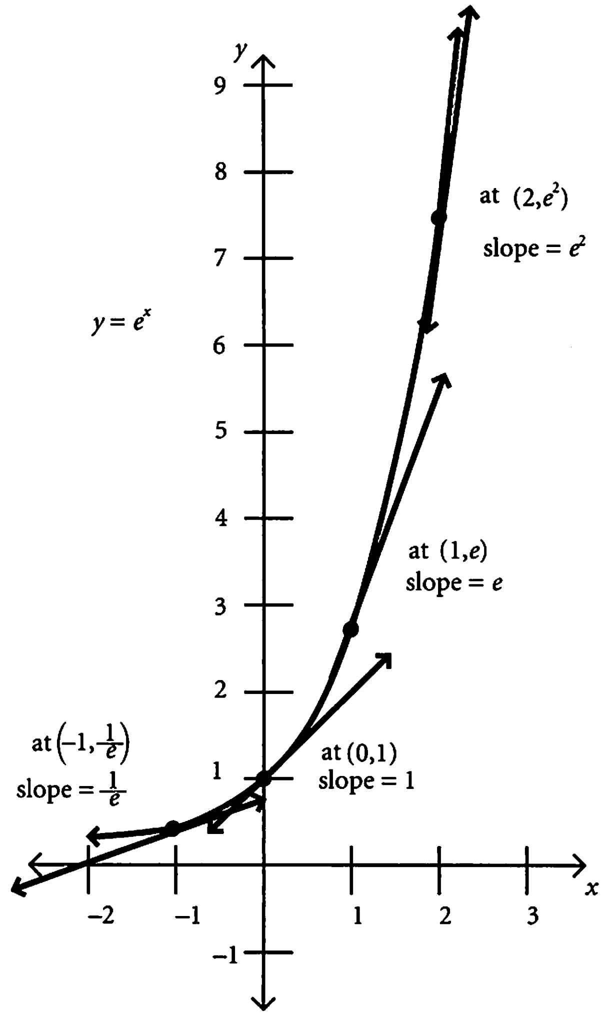

Figure 8.5

As for the inverse function $\ln(x)$, the natural logarithm is written as follows:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Lesson 11 will contain a proof for this.

Examples

Differentiate the following.

$f(x) = 4e^x$

$y = 10e^x + 10$

$g(t) = 3e^t + 2\ln(t)$

$y = 8\ln(u) - e^u + 7u$

Solutions

$f'(x) = 4e^x$

$\frac{dy}{dx} = 10e^x$

$g'(t) = 3e^t + \frac{2}{t}$

$\frac{dy}{du} = \frac{8}{u} - e^u + 7$

▶ Practice

Differentiate the following.

19. $f(x) = 1 + x + x^2 + x^3 + e^x$

20. $g(t) = 12\ln(t) + t^2 + 4$

21. $y = \cos(x) - 10e^x + 8x$

22. $h(x) = \sqrt{x} - 8\ln(x)$

23. $k(u) = 3x^{\frac{5}{2}} + 5e^x + 11$

24. Find the second derivative of $f(x) = e^x + \ln(x)$.

25. Find the 100th derivative of $g(x) = 3e^x$.

26. What is the slope of the tangent line to $f(x) = \ln(x)$ at $x = 10$?